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The quantum theory of measurement within dynamical reduction models

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Abstract

We analyse in mathematical detail, within the framework of the QMUPL model of spontaneous wavefunction collapse, the von Neumann measurement scheme for the measurement of a $1/2$ spin particle. We prove that, according to the equation of the model, (i) throughout the whole measurement process, the pointer of the measuring device is always perfectly well localized in space; (ii) the probabilities for the possible outcomes are distributed in agreement with the Born probability rule; (iii) at the end of the measurement the state of the microscopic system has collapsed to the eigenstate corresponding to the measured eigenvalue. This analysis shows rigorously how dynamical reduction models provide a consistent solution to the measurement problem of quantum mechanics.

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1. Introduction

In standard textbooks on quantum mechanics, e.g. in [1], one can find the following axioms defining the quantum theory:

Axiom 1 (states). A Hilbert $\mathcal{H}$ space is associated with each physical system and the state of the system is represented by a vector $|\psi\rangle$ in $\mathcal{H}$. (In the following, we will always assume vectors to be normalized.)

Axiom 2 (observables). To any observable quantity of the system is associated with a self-adjoint operator in $\mathcal{H}$. The only possible outcomes of a measurement of an observable are the eingenvalues of the associated operator.

Axiom 3 (Schrödinger equation). Given $|\psi_0\rangle$ the state of the system at an initial time $t_0 = 0$, its state at any subsequent time t is represented by $|\psi_t\rangle$, which is the solution of the

Schrödinger equation:

$$i\hbar \frac{d}{dt}|\psi_t\rangle = H|\psi_t\rangle, \tag{1}$$

for the given initial condition; the self-adjoint operator H is the Hamiltonian operator associated with the system.

Axiom 4 (Born rule). Let $|\psi\rangle$ be the vector describing the state of the system at a given time; then the probability that the outcome of a measurement of an observable $\mathcal{A}$ at that time is one of the values a_n belonging to the spectrum of A, is given by the Born probability rule:

$$\mathbb{P}[a_n] = \langle\psi|P_n|\psi\rangle, \tag{2}$$

where P_n is the projection operator associated with the eigenmanifold of the operator A corresponding to the eigenvalue a_n.

Axiom 5 (wave-packet reduction). At the end of a measurement process the state of the system changes according to the rule:

$$|\psi\rangle \xrightarrow[\text{[after measurement]}]{} \frac{P_n|\psi\rangle}{\|P_n|\psi\rangle\|}, \tag{3}$$

where P_n is the projection operator associated with the outcome a_n of the measurement.

As well known, the last axiom gives rise to the measurement problem in quantum mechanics, because of which the theory, as it stands, cannot be considered a consistent description of physical phenomena. Many tentative solutions have been suggested, among which dynamical reduction models are one of the few promising proposals; their general structure has been already fully described in the past literature [2, 3]; here we limit ourselves to list the axioms defining them (at the non-relativistic level):

Axiom A (states). A Hilbert $\mathcal{H}$ space is associated with each physical system and the state of the system is represented by a (normalized) vector $|\psi\rangle$ in $\mathcal{H}$.

Axiom B evolution (continuous version). Given the system initially in a state described by the vector $|\psi_0\rangle$, its state at any subsequent time t is represented by $|\psi_t\rangle$, which solves the following stochastically modified Schrödinger equation:

$$d|\psi_t\rangle = \left[-\frac{i}{\hbar}H\,dt + \sqrt{\lambda}(A - \langle A\rangle_t)\,dW_t - \frac{\lambda}{2}(A - \langle A\rangle_t)^2\,dt\right]|\psi_t\rangle, \tag{4}$$

where W_t is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, while $\langle A\rangle_t \equiv \langle\psi_t|A|\psi_t\rangle$ is the quantum average value of the operator A, which is a suitably chosen (in this case, self-adjoint) operator; λ is a positive constant controlling the strength of the collapse[4].

Axiom C (ontology). Let $\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \equiv \langle\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N|\psi\rangle$ the wavefunction for a system of N particles (which for simplicity we take to be scalar) in configuration space. Then

$$\mu_t^{(n)}(\mathbf{x}_n) \equiv m_n \int d^3x_1 \ldots d^3x_{n-1}\, d^2x_{n+1} \ldots d^3x_N |\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)|^2 \tag{5}$$

represents the *density of mass*[5] of the nth particle of the system, to which a total mass m_n is associated [4, 5].

[4] Equation (4) can be generalized in different directions [2, 3]; however, its general structure has to be preserved in order for the model to provide a solution to the measurement problem.

[5] In the subsequent sections, for simplicity's sake, we will not make reference to the mass density function anymore, but we will only keep track of the evolution of the wavefunction; however it should be clear that, in order to be fully rigorous, all statements about the properties of physical systems should be phrased in terms of their mass–density distribution, not in terms of the wavefunction.

Axiom A is equal to 1, while B replaces 3. Indeed, B embodies 3, meaning with this that a sensible choice for λ and A can be made such that equation (4) practically reduces to equation (1) when a microscopic quantum system is taken into account (see [6] for a recent and exhaustive review of the subject). The remarkable property of collapse models is that also the other axioms of quantum mechanics derive from B (and C, of course): the aim of this paper is to show how axioms 4 and 5 derive from axiom B, while in a future paper we will discuss how axiom 2 also derives from B. To be more precise, following the previous work of [7], we here analyse in mathematical detail, within the framework of a specific dynamical reduction model, a von Neumann type of measurement scheme, in which a microscopic system interacts with a macroscopic apparatus devised in such a way to measure one or more properties of the micro-system. We will show, giving also precise estimates, that

(1) whichever the initial state of the microscopic system, throughout the entire measurement process the apparatus has a definite position in space, its wavefunction being always extremely well localized;
(2) the only possible outcomes correspond to those given by standard quantum mechanics, with probability almost equal to 1;
(3) the probability of getting a certain outcome is given by the *Born probability rule* within an exceedingly high degree of approximation;
(4) after the measurement, the state vector of the microscopic system collapses to a state which practically coincides with the eigenstate of the measured observable, corresponding to the eigenvalue which has been observed.

Needless to say, these properties were already known since very long time, and indeed they represent the very motivation behind the original GRW model [8] and its subsequent generalizations, and the reason for its success; our goal here is to derive them in a rigorous way from the equations of a specific model of wavefunction collapse.

The paper is organized as follows. In section 2 we will introduce the measurement model our analysis is based upon, and we will discuss its physical features. In section 3 we will study the special case in which the microscopic system has been prepared in an eigenstate of the operator associated with the observable the model is devised to measure, while in section 4 we will analyse in full detail the case of an arbitrary initial state. In the concluding section 5 we will summarize the features of our model and draw our final conclusions.

2. The measurement model

We begin our discussion by presenting the measurement model we will use in the following sections: the setup consists of a microscopic system $\mathcal{S}$ interacting with a macroscopic system $\mathcal{A}$ which acts like a measuring apparatus; both systems are described in quantum mechanical terms. Here below we give the details.

2.1. The microscopic system

We consider a single measurement process, in which the experimenter is able to distinguish among a *finite* set of outcomes. Accordingly, we assume that the microscopic system $\mathcal{S}$ can be described, for what concerns the measurement process, by a finite-dimensional complex Hilbert space. For the sake of simplicity, and without loss of generality, we can consider the simplest case: $\mathcal{H}_{\mathcal{S}} = \mathbb{C}^2$, because the generalization of what follows to $\mathbb{C}^n$ is quite straightforward. Since the most general self-adjoint operator O acting on $\mathbb{C}^2$ can be written as

$$O = o_+|+\rangle\langle+| + o_-|-\rangle\langle-|, \tag{6}$$

where $|+\rangle$ and $|-\rangle$ are the eigenstates of O, while o_+ and o_- are its two real eigenvalues, for definiteness and with no loss of generality, in what follows we will take $o_\pm = \pm\hbar/2$ and O to be the z-component of the spin, S_z, of a 1/2 spin particle.

2.2. The measuring apparatus

We take the following model for the measuring apparatus $\mathcal{A}$, which is general enough to describe all interesting physical situations: we assume that the apparatus consists of a fixed part plus a pointer moving along a graduate scale, in such a way that different positions of the pointer along the scale correspond to different possible outcomes of the measurement. To simplify the analysis, we study the evolution of the centre of mass of the pointer only, and disregard all other macroscopic and microscopic degrees of freedom; accordingly, the pointer will be treated like a macroscopic quantum particle of mass m moving in one dimension only, whose state space is described by the Hilbert space $\mathcal{H}_\mathcal{A} = \mathrm{L}^2(\mathbb{R})$.

2.3. The dynamics

We assume that the pointer of $\mathcal{A}$ undergoes a spontaneous collapse mechanism according to the Quantum Mechanics with Universal Position Localization (QMUPL) model first introduced in [9] and subsequently analysed in [10] (see also references therein), while the microscopic system $\mathcal{S}$ evolves according to the standard Schrödinger equation, since, as typical of dynamical reduction models, the stochastic collapse terms have very little effects on microscopic quantum systems. Accordingly, we take for the evolution equation of the composite $\mathcal{S}+\mathcal{A}$ system the following stochastic differential equation[6] (SDE) defined in the Hilbert space $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{A} := \mathbb{C}^2 \otimes \mathrm{L}^2(\mathbb{R})$:

$$\mathrm{d}\,|\Psi_t\rangle = \left[-\frac{\mathrm{i}}{\hbar}H_t\,\mathrm{d}t + \sqrt{\lambda}\,(q - \langle q\rangle_t)\,\mathrm{d}W_t - \frac{\lambda}{2}\,(q - \langle q\rangle_t)^2\,\mathrm{d}t\right]|\Psi_t\rangle\,, \tag{7}$$

which is precisely of the form (4), where H_t is the (time dependent) standard Hamiltonian operator of the composite system, and q is the position operator associated with the centre of mass of the pointer.[7] In the following we will use capital letters (Ψ, Φ, ...) to denote a state vector for the composite $\mathcal{S}+\mathcal{A}$ system, and lower case letters ($\psi, \phi, \ldots$) to denote a state vector referring to the pointer alone.

As discussed in [10] we take for the constant λ appearing in (7)

$$\lambda \simeq \frac{m}{m_0}\lambda_0, \tag{8}$$

with $m_0 \simeq 1.7 \times 10^{-27}$ kg being a reference mass of order of a nucleon mass and $\lambda_0 \simeq 10^{-2}\ \mathrm{m}^{-2}\,\mathrm{s}^{-1}$. For definiteness, let us consider a pointer of mass $m = 1$ g (i.e., a pointer made of an Avogadro number of nucleons), and let us define, for later convenience, the quantities

$$\omega := 2\sqrt{\frac{\hbar\lambda}{m}} \simeq 5.0 \times 10^{-5}\ \mathrm{s}^{-1} \qquad \text{and} \qquad \sigma_q := \sqrt{\frac{\hbar}{m\omega}} \simeq 4.6 \times 10^{-14}\ \mathrm{m}. \tag{9}$$

We take the Hamiltonian H_t to be of the form $H_t = H_\mathcal{S} + H_\mathcal{A} + H_{\mathrm{INT}}$. The first term is the quantum Hamiltonian for the microscopic system: we assume that the time scale of the free evolution of the microscopic system is much larger than the characteristic time scale of the experiment ('instantaneous measurement' assumption); accordingly we take $H_\mathcal{S}$ to be the null

[6] See [11] and [12] for theorems on the existence and uniqueness of solution for this type of equation.

[7] Thus, strictly speaking, we should write $I_\mathcal{S} \otimes q$ for the position operator for the pointer, where $I_\mathcal{S}$ is the identity operator in $\mathcal{H}_\mathcal{S}$. We avoid such a way of writing, when no confusion arises.

operator. The second term is the quantum Hamiltonian of the pointer, which we take equal to that of a non-relativistic free quantum particle of mass m: $H_{\mathcal{A}} = p^2/(2m)$, where p is the momentum operator.

Finally, we assume the interaction term H_{INT} between the two systems to be of the von Neumann type, and devised in such a way to measure the spin operator S_z:

$$H_{\mathrm{INT}}(t) = \kappa \Delta_t^T S_z \otimes p, \tag{10}$$

where κ is a coupling constant and $\Delta^T : t \mapsto \Delta_t^T$ is a T-normalized,[8] non-negative, real valued, function of time, identically equal to zero outside a given interval of the form $(t_0, t_0 + T)$, i.e., outside the time interval of length T, say $T = 1$ s, during which the experiment takes place; we choose the time origin in such a way that the experiment begins at $t_0 = 0$ s. As it is well known, H_{INT} generates the following type of evolution, depending on the initial state of the micro-system $\mathcal{S}$:

$$[c_+|+\rangle + c_-|-\rangle] \otimes |\phi_0\rangle \mapsto c_+|+\rangle \otimes |\phi_+\rangle + c_-|-\rangle \otimes |\phi_-\rangle, \tag{11}$$

where $|\phi_\pm\rangle$ are final pointer states spatially translated with respect to the initial state $\left|\phi_0\right\rangle$ by the quantity $\pm(\hbar/2)\kappa T$.

The strength of the coupling constant κ has to be chosen in such a way that the distance $\hbar\kappa T$ between the initial state $|\phi_0\rangle$ of the pointer and any of the two final states $|\phi_\pm\rangle$ is macroscopic; for definiteness, let us choose $\hbar\kappa = 1$ cm s^{-1}, so that $\hbar\kappa T = 1$ cm.

2.4. The initial state

We take the initial states of the microscopic system $\mathcal{S}$ and of the macroscopic apparatus $\mathcal{A}$ to be completely uncorrelated, as it is customary and appropriate for the description of a measurement process. Accordingly, we assume the initial state of the total system $\mathcal{S} + \mathcal{A}$ to be

$$[c_+|+\rangle + c_-|-\rangle] \otimes |\phi_0\rangle, \tag{12}$$

where $|\phi_0\rangle$ describes the 'ready' state of the macroscopic apparatus $\mathcal{A}$.

Regarding the initial state $|\phi_0\rangle$ of the pointer, some considerations have to be done. In [10] it has been shown that, according to equation (7), the wavefunction for the centre of mass of an isolated quantum system reaches asymptotically (and very rapidly, for a macro-object) a Gaussian state of the form

$$\phi_t^{\mathrm{G}}(x) = \sqrt[4]{\frac{1}{2\pi\sigma_q^2}} \exp\left[-\frac{1-i}{4\sigma_q^2}(x - \bar{x}_t)^2 + i\bar{k}_t x\right], \tag{13}$$

(modulo a time-dependent global phase factor) with σ_q defined as in equation (9). For later reference, let us observe that the dispersion of the Gaussian function of equation (13) in momentum space is

$$\sigma_p = \frac{\hbar}{\sqrt{2}\sigma_q} \simeq 1.6 \times 10^{-21} \text{ kg m s}^{-1}, \tag{14}$$

quite close to the minimum allowed by Heisenberg's uncertainty relation, and that the centres of ϕ_t^{G} in position and momentum space are given by $\bar{x}_t$ and $\hbar\bar{k}_t$, respectively.

[8] By a T-normalized function, we just mean

$$\int_{-\infty}^{+\infty} \Delta_t^T \, \mathrm{d}t = \int_{t_0}^{t_0+T} \Delta_t^T \, \mathrm{d}t = T.$$

Note that Δ_t^T depends also on the initial time t_0; we will avoid to indicate it explicitly, when non-confusion arises.

In our measurement model, we assume that the pointer is isolated for the time prior to the experiment; during this time, as shown in the past literature, its wavefunction converges rapidly towards a state close to (13), which we therefore assume to be the initial state of the pointer. To summarize, we take as the initial state of the composite system $\mathcal{S}+\mathcal{A}$ the ket

$$|\Psi_0\rangle = [c_+|+\rangle + c_-|-\rangle] \otimes |\mathrm{G}, 0\rangle, \tag{15}$$

where $\langle x|\mathrm{G}, 0\rangle$ is of the form (13). We choose the natural reference frame where the pointer is initially at rest, so that $\bar{k}_0 = 0\ \mathrm{m}^{-1}$, with the origin set-up in such a way that $\bar{x}_0 = 0\ \mathrm{m}$.

3. Measurement of an eigenstate

We begin our study of the model by looking for the solution of equation (7) satisfying the initial condition

$$\left|\Psi_0^{\pm}\right\rangle = |\pm\rangle \otimes |\mathrm{G}, 0\rangle\,, \tag{16}$$

where the symbol $\pm$ means that the state $|\pm\rangle$ is *either* $|+\rangle$ *or* $|-\rangle$, i.e., an eigenstate of the operator S_z. We will show that, in this special case, the state of the microscopic system does not change in time, while the pointer moves along the scale so to give the correct outcome of the measurement.

3.1. The linear equation

Following the standard procedure outlined e.g. in [10] we pass from the nonlinear equation (7) to the associated equation

$$\mathrm{d}\,|\Phi_t\rangle = \left[-\frac{\mathrm{i}}{\hbar}H_t\,\mathrm{d}t + \sqrt{\lambda}q\,\mathrm{d}\xi_t - \frac{\lambda}{2}q^2\,\mathrm{d}t\right]|\Phi_t\rangle\,, \tag{17}$$

where ξ_t is a standard Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, the measure $\mathbb{Q}$ being a new probability measure chosen in such a way that the probability measure $\mathbb{P}$ previously introduced is generated from $\mathbb{Q}$ by the martingale $\|\Phi_t\|^2$. The Wiener process ξ_t of equation (17) is related to the Wiener process W_t of equation (7) via Girsonov's rule [13]:

$$\mathrm{d}W_t = \mathrm{d}\xi_t - 2\sqrt{\lambda}\langle q\rangle_t\,\mathrm{d}t. \tag{18}$$

It is easy to prove that a vector of the form

$$\left|\Phi_t^{\pm}\right\rangle = |\pm\rangle \otimes \left|\phi_t^{\pm}\right\rangle \tag{19}$$

solves equation (17), for the initial condition (16), if the wavefunction $\phi_t^{\pm}(x) := \left\langle x\middle|\phi_t^{\pm}\right\rangle$ solves the following linear SDE, which involves the apparatus degrees of freedom alone:

$$\mathrm{d}\phi_t^{\pm}(x) = \left[\left(\frac{\mathrm{i}\hbar}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} \mp \frac{\hbar\kappa}{2}\Delta_t^T\frac{\mathrm{d}}{\mathrm{d}x}\right)\mathrm{d}t + \sqrt{\lambda}x\,\mathrm{d}\xi_t - \frac{\lambda}{2}x^2\,\mathrm{d}t\right]\phi_t^{\pm}(x). \tag{20}$$

3.2. The solution and its properties

The solution of equation (20) for the given initial condition is the Gaussian wavefunction

$$\phi_t^{\pm}(x) = \exp\left[-\alpha_t\left(x - \bar{x}_t^{\pm}\right)^2 + \mathrm{i}\bar{k}_t^{\pm}x + \gamma_t^{\pm} + \mathrm{i}\theta_t^{\pm}\right], \tag{21}$$

whose parameters $\alpha_t \in \mathbb{C}$, and $\bar{x}_t^{\pm}, \bar{k}_t^{\pm}, \gamma_t^{\pm}, \theta_t^{\pm} \in \mathbb{R}$ (of obvious meaning) satisfy the following system of SDE [10]:

$$\mathrm{d}\alpha_t = \left(\lambda - \frac{2\mathrm{i}\hbar}{m}\alpha_t^2\right)\mathrm{d}t \tag{22}$$

$$\mathrm{d}\bar{x}_t^{\pm} = \left(\frac{\hbar}{m}\bar{k}_t^{\pm} \pm \frac{\hbar}{2}\kappa\Delta_t^T\right)\mathrm{d}t + \frac{\sqrt{\lambda}}{2\alpha_t^{\mathrm{R}}}\left\{\mathrm{d}\xi_t - 2\sqrt{\lambda}\bar{x}_t^{\pm}\,\mathrm{d}t\right\} \tag{23}$$

$$\mathrm{d}\bar{k}_t^{\pm} = -\sqrt{\lambda}\frac{\alpha_t^{\mathrm{I}}}{\alpha_t^{\mathrm{R}}}\left\{\mathrm{d}\xi_t - 2\sqrt{\lambda}\bar{x}_t^{\pm}\,\mathrm{d}t\right\} \tag{24}$$

$$\mathrm{d}\gamma_t^{\pm} = \left(\lambda\left(\bar{x}_t^{\pm}\right)^2 + \frac{\hbar}{m}\alpha_t^{\mathrm{I}} + \frac{\lambda}{4\alpha_t^{\mathrm{R}}}\right)\mathrm{d}t + \sqrt{\lambda}\bar{x}_t^{\pm}\left\{\mathrm{d}\xi_t - 2\sqrt{\lambda}\bar{x}_t^{\pm}\,\mathrm{d}t\right\} \tag{25}$$

$$\mathrm{d}\theta_t^{\pm} = \left(-\frac{\hbar}{2m}\left(\bar{k}_t^{\pm}\right)^2 - \frac{\hbar}{m}\alpha_t^{\mathrm{R}} + \frac{\lambda\alpha_t^{\mathrm{I}}}{4\left(\alpha_t^{\mathrm{R}}\right)^2} \mp \frac{\hbar\kappa}{2}\Delta_T(t)\bar{k}_t^{\pm}\right)\mathrm{d}t + \sqrt{\lambda}\frac{\alpha_t^{\mathrm{I}}}{\alpha_t^{\mathrm{R}}}\bar{x}_t^{\pm}\left\{\mathrm{d}\xi_t - 2\sqrt{\lambda}\bar{x}_t^{\pm}\,\mathrm{d}t\right\}, \tag{26}$$

where we have denoted by z^{R} (z^{I}) the real (imaginary) part of the complex number z.

Equations (25) and (26) are of no particular interest in this simple situation, because they just describe the time evolution of the irrelevant norm and global phase of the Gaussian solution. Equation (22) is independent of equations (23)–(26); it is deterministic, and easily solved by separation of variables:

$$\alpha_t = \frac{1-i}{4\sigma_q^2}\tanh\left(\frac{\omega t}{1-i} + c_0\right), \tag{27}$$

where c_0 sets the initial condition. Equation (27) determines the time evolution of the spread in position and momentum of the Gaussian wavefunction. In our case, given the initial condition $\alpha_0 = (1-\mathrm{i})/4\sigma_q^2$, it follows that $\alpha_t \equiv (1-\mathrm{i})/4\sigma_q^2$ for all times (i.e., we can set $\alpha_t^{\mathrm{R}} \equiv 1/4\sigma_q^2$ and $\alpha_t^{\mathrm{I}} \equiv -1/4\sigma_q^2$ in equations (23)–(26)): as expected, the position and momentum spreads of the wavefunction of the pointer do not change in time, and remain identically equal to, respectively, σ_q and σ_p.

For what concerns equations (23) and (24), which do not depend on equations (25) and (26), their solution describes the time evolution of the mean value in position and momentum of the Gaussian wavefunction. We have to characterize the stochastic properties of the solution with respect to the physical probability measure $\mathbb{P}$, i.e. we need to go back to the original noise W_t via Girsanov's rule (18), which in this case is very easy, because for a wavefunction like (21) we simply have $\langle q\rangle_t = \bar{x}_t^{\pm}$ for all times, so that all we have to do is to write $\mathrm{d}W_t$ in place of the curly braces $\{\cdots\}$ in equations (23) and (24):

$$\mathrm{d}\bar{x}_t^{\pm} = \left(\frac{\hbar}{m}\bar{k}_t^{\pm} \pm \frac{\hbar\kappa}{2}\Delta_t^T\right)\mathrm{d}t + \sigma_q\sqrt{\omega}\,\mathrm{d}W_t \tag{28}$$

$$\mathrm{d}\bar{k}_t^{\pm} = \frac{\sigma_p}{\sqrt{2}\hbar}\sqrt{\omega}\,\mathrm{d}W_t, \tag{29}$$

where we have also taken into account that $\alpha_t = (1-\mathrm{i})/4\sigma_q^2$ for any $t \geqslant 0$. Let us call $\left|\Psi_t^{\pm}\right\rangle$ the normalized physical solutions: $\left|\Psi_t^{\pm}\right\rangle \equiv \left|\Phi_t^{\pm}\right\rangle / \left\|\left|\Phi_t^{\pm}\right\rangle\right\|$, with $\left|\Phi_t^{\pm}\right\rangle$ given by equation (19); taking also into account that $\bar{x}_0 = 0$ m, and $\bar{k}_0 = 0\ \mathrm{m}^{-1}$, we find the following results.

(1) According to equation (29), the average value of the peak of the Gaussian wavefunction in momentum space, $\langle p\rangle_t^{\pm} := \left\langle\Psi_t^{\pm}\right|p\left|\Psi_t^{\pm}\right\rangle \equiv \hbar\bar{k}_t^{\pm}$, does not evolve in time:

$$\mathbb{E}_{\mathbb{P}}\left[\langle p\rangle_t^{\pm}\right] = \bar{p}_0^{\pm} = 0\ \mathrm{kg\ m\ s}^{-1}. \tag{30}$$

(2) By equations (28) and (29), the average value of the peak in position space $\langle q\rangle_t^{\pm} := \langle\Psi_t^{\pm}|q|\Psi_t^{\pm}\rangle \equiv \bar{x}_t^{\pm}$ of the Gaussian wavefunction evolves in time according to

$$\mathbb{E}_{\mathbb{P}}\big[\langle q\rangle_t^{\pm}\big] = \pm\frac{\hbar\kappa}{2}\int_0^t \Delta_{t'}^T\,\mathrm{d}t'. \tag{31}$$

Equation (31) shows that, when the measurement begins, $\mathbb{E}_{\mathbb{P}}[\langle q_t^{\pm}\rangle]$ moves towards the right or the left according to the initial state of the microscopic system $\mathcal{S}$, reaching the final value (at the end of the measurement)

$$\mathbb{E}_{\mathbb{P}}\big[\langle q\rangle_t^{\pm}\big] = \pm\frac{\hbar\kappa T}{2} = \pm 0.5\ \text{cm} \qquad \text{for } t \geqslant T. \tag{32}$$

(3) The variance $\mathbb{V}_{\mathbb{P}}[\langle q\rangle_t^{\pm}] \equiv \mathbb{E}_{\mathbb{P}}[\langle q\rangle_t^{\pm} - \mathbb{E}_{\mathbb{P}}[\langle q\rangle_t^{\pm}]]^2$ associated with the motion of $\langle q\rangle_t^{\pm}$ is equal to the variance computed in [10] (see section VII B), which, for $\alpha_t \equiv (1-\mathrm{i})/4\sigma_q^2$, is given by[9]:

$$\mathbb{V}_{\mathbb{P}}\big[\langle q\rangle_t^{\pm}\big] = \sigma_q^2\left[\omega t + \frac{(\omega t)^2}{2} + \frac{(\omega t)^3}{12}\right]; \tag{33}$$

with our choices for the parameters, we have $\mathbb{V}_{\mathbb{P}}\big[\langle q\rangle_t^{\pm}\big] \leqslant \mathbb{V}_{\mathbb{P}}\big[\langle q\rangle_T^{\pm}\big] \simeq 1.1\times 10^{-31}\ \mathrm{m}^2$, for any $t \leqslant T$.

From the above results we can derive the following important conclusions:

- Due to the smallness of its variance, the motion of the peak $\langle q\rangle_t^{\pm}$ of the Gaussian wavefunction for the c.m. of the pointer is practically *deterministic* and equivalent to the motion of $\mathbb{E}_{\mathbb{P}}\big[\langle q\rangle_t^{\pm}\big]$, the fluctuations around the mean being so tiny that they can be safely ignored. E.g., the probability for $\langle q\rangle_t^{\pm}$ to lie outside an interval of width Δ centred in $\mathbb{E}_{\mathbb{P}}\big[\langle q\rangle_t^{\pm}\big]$ can be estimated by using Čebičev's inequality; for $\Delta = 10^{-5}$ cm, we have

$$\mathbb{P}\big[\big|\langle q\rangle_t^{\pm} - \mathbb{E}_{\mathbb{P}}\big[\langle q\rangle_t^{\pm}\big]\big| \geqslant \Delta/2\big] \leqslant 4\frac{\mathbb{V}_{\mathbb{P}}\big[\langle q_T^{\pm}\rangle\big]}{\Delta^2} \simeq 4.2\times 10^{-17}, \qquad \text{for any } t \leqslant T, \tag{34}$$

a vanishingly small probability.

- As such, and because of equation (31), the peak $\langle q\rangle_t^{\pm}$ evolves in time as follows:

$$\langle q\rangle_t^{\pm} = \begin{cases} \pm\dfrac{\hbar\kappa}{2}\displaystyle\int_0^t \Delta_{t'}^T\,\mathrm{d}t' & t \leqslant T \quad (+\text{ negligible fluctuations}) \\ \pm\dfrac{\hbar\kappa T}{2} = \pm 0.5\ \text{cm} & t \geqslant T \quad (+\text{ negligible fluctuations}) \end{cases} \tag{35}$$

This means that, according to the initial state of the micro-system, the pointer moves in a practically deterministic way either towards left or towards right, with respect to the initial ready state, displaying in this way the outcome of the measurement.

- During the measurement, the state of the micro-system does not change.

This is precisely the expected behaviour both for the microscopic system as well as for the macroscopic pointer, when the initial state is given by (19), for an ideal measurement scheme as the one of von Neumann here analysed.

[9] We correct in this way a typo contained in equation (93) of [10].

4. Measurement of a superposition

Let us now consider the general case where the initial state $|s_i\rangle$ of the microscopic system $\mathcal{S}$ is not an eigenstate of S_z, but a superposition of eigenstates of the form

$$|s_i\rangle = c_+|+\rangle + c_-|-\rangle \qquad (|c_+|^2 + |c_-|^2 = 1); \tag{36}$$

the global initial condition for the micro-system and the apparatus then is

$$|\Psi_0\rangle = [c_+|+\rangle + c_-|-\rangle] \otimes |G, 0\rangle\,. \tag{37}$$

As in the preceding subsection, we first solve the linear equation, and next move to the nonlinear one. Due to the linearity of equation (17), its solution, with the given initial condition (37), is

$$|\Phi_t\rangle = |+\rangle \otimes \left|\phi_t^+\right\rangle + |-\rangle \otimes \left|\phi_t^-\right\rangle, \tag{38}$$

where the wavefunctions $\left|\phi_t^+\right\rangle$ and $\left|\phi_t^-\right\rangle$, in the position representation, are of the form (21) and the parameters α_t, $\bar{x}_t^\pm$, $\bar{k}_t^\pm$, $\gamma_t^\pm$, $\theta_t^\pm$ solve equations (22)–(26), with the obvious choice of sign and with initial conditions:

$$\alpha_0 = \frac{1-i}{4\sigma_q^2}, \qquad \bar{x}_0^\pm = 0\ \mathrm{m}, \qquad \bar{k}_0^\pm = 0\ \mathrm{m}^{-1}, \qquad \gamma_0^\pm = \ln|c_\pm|, \qquad \theta_0^\pm = \mathrm{Arg}[c_\pm] \tag{39}$$

(of course we now assume that $c_\pm \neq 0$).

Since the time evolution of the parameters α_t is governed by equation (22) which, as we have already remarked, is deterministic and decoupled from the evolution equations (23)–(26) for the remaining parameters, we observe first of all that the evolution of the spreads in position and momentum of the two Gaussian functions $\left|\phi_t^+\right\rangle$ and $\left|\phi_t^-\right\rangle$ does not change with respect to the case analysed in the previous section. Accordingly, we have $\alpha_t \equiv (1-i)/4\sigma_q^2$ for all times, so that the spreads of the two wavefunctions do not evolve and remain identically equal to the asymptotic values σ_q and σ_p.

4.1. The deterministic evolution of the distances in position and momentum between the two Gaussian components

Contrary to the preceding case, moving from the solution of the linear equation to the solution of the nonlinear one is not immediate, since Girsanov's rule (18) involves the quantum average $\langle q\rangle_t$, which in this case turns out not to be a trivial function of the parameters controlling the two Gaussian components; namely, one finds that[10]

$$\langle q\rangle_t = \frac{\bar{x}_t^+\,\mathrm{e}^{2\gamma_t^+} + \bar{x}_t^-\,\mathrm{e}^{2\gamma_t^-}}{\mathrm{e}^{2\gamma_t^+} + \mathrm{e}^{2\gamma_t^-}}. \tag{40}$$

Of course, this is an entirely expected difficulty, due to the essential nonlinearity inherent to collapse models; to proceed in the analysis of the problem, it is convenient first of all to analyse the evolution of the distance between the maxima of the two Gaussian functions $\left|\phi_t^+\right\rangle$ and $\left|\phi_t^-\right\rangle$, both in position as well as in momentum space, and subsequently of their relative weights.

Following the path outlined in [10], let us consider the differences $X_t := \bar{x}_t^+ - \bar{x}_t^-$ and $K_t := \bar{k}_t^+ - \bar{k}_t^-$, which express at each instant the distance in position and (modulus $\hbar$) momentum space between the centres of the two Gaussian functions $\left|\phi_t^+\right\rangle$ and $\left|\phi_t^-\right\rangle$. From

[10] In equation (40) no contribution comes from the overlapping between the two Gaussian components, since each component is coupled to one of the two orthogonal spin state $|\pm\rangle$, which make the 'off-diagonal' terms of the scalar product vanish.

equations (23) and (24), keeping in mind that in our case we have $\alpha_t^{\mathrm{R}} \equiv 1/4\sigma_q^2$ and $\alpha_t^{\mathrm{I}} \equiv -1/4\sigma_q^2$ for all times, we get the following *deterministic* system for X_t and K_t:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} X_t \\ K_t \end{bmatrix} = \begin{bmatrix} -\omega & \hbar/m \\ -2\lambda & 0 \end{bmatrix}\begin{bmatrix} X_t \\ K_t \end{bmatrix} + \begin{bmatrix} \hbar\kappa\Delta_t^T \\ 0 \end{bmatrix}; \tag{41}$$

since it does not depend on the noise, it is insensitive to the change of measure and holds true also for the nonlinear equation (7).

The solution of the above system depends of course on the specific choice for the function Δ_t^T; a simple reasonable choice is the following:

$$\Delta_t^T = \begin{cases} 1 & t \in [0, T] \\ 0 & \text{else,} \end{cases} \tag{42}$$

which, according to equation (35), means that, with restriction to the situation analysed in the previous section, during the measurement the pointer moves at a constant speed either towards the left or towards the right, depending on the initial state of the micro-system. According to this choice, X_t, given the initial condition $X_0 = 0$ m, evolves in time as follows:

$$X_t = \begin{cases} \dfrac{2\hbar\kappa}{\omega}\,\mathrm{e}^{-\omega t/2}\sin\dfrac{\omega}{2}t & \text{for } 0 \leqslant t \leqslant T, \\ \dfrac{2\hbar\kappa}{\omega}\,\mathrm{e}^{-\omega t/2}\left[\sin\dfrac{\omega}{2}t - \mathrm{e}^{\omega T/2}\sin\dfrac{\omega}{2}(t-T)\right] & \text{for } t \geqslant T. \end{cases} \tag{43}$$

Since $\omega^{-1} \simeq 2.0 \times 10^4$ s is a very long time compared to the measurement time, we can meaningfully expand equation (43) to first order in ωt:

$$X_t \simeq \begin{cases} \hbar\kappa t & \text{for } 0 \leqslant t \leqslant T = (\hbar\kappa)^{-1} = 1 \text{ s}, \\ 1 \text{ cm} & \text{for } T \leqslant t \ll \omega^{-1} \simeq 2.0 \times 10^4 \text{ s}. \end{cases} \tag{44}$$

As we see, the distance between the two peaks increases almost linearly in time, reaching its maximum (1 cm) at the end of the measurement process, as predicted by the standard Schrödinger equation; after this time, their distance remains practically unaltered for extremely long times, and only for $t \simeq 2.0 \times 10^4$ s it starts slowly decreasing, eventually going to 0. Note that such a behaviour, being determined by ω, does *not* depend on the mass of the pointer, thus a larger pointer will not change the situation. The moral is that X_t behaves as if the reduction mechanism were not present (as if $\lambda_0 = 0$) so we have to look for the collapse somewhere else.

As we shall discuss in the next subsection, the collapse occurs because, in a very short time, the weight of one of the two Gaussian wavefunctions ($|\phi_t^+\rangle$ or $|\phi_t^-\rangle$) becomes much smaller than the weight of the other component; this implies that, when the normalization of the whole state is taken into account, one of the two components practically disappears, and only the other one survives, the one which sets the outcome of the experiment. Of course, this process is random and, as we shall prove, it occurs with a probability almost equivalent to the Born probability rule.

4.2. The evolution equation governing the relative weight of the two Gaussian components

The relative damping between the two Gaussian components of equation (38) is measured by the stochastic process $\Gamma_t = \gamma_t^+ - \gamma_t^-$: if, at a certain time t, it occurs that $\Gamma_t \gg 1$, it means that at the end of the experiment $|\phi_t^-\rangle$ is suppressed with respect to $|\phi_t^+\rangle$, so that the initial state (37) practically evolves to $|+\rangle \otimes |\psi_t^+\rangle$ (remember that $|\psi_t^\pm\rangle = |\phi_t^\pm\rangle / \||\phi_t^\pm\rangle\|$ are the normalized

states); the opposite happens if $\Gamma_t \ll -1$. To be quantitative, let us introduce a conveniently large collapse parameter, say 35, and the following definition[11]:

Definition. *The superposition (38) is suppressed when* $|\Gamma_t| \geqslant 35$*, i.e., when either* $\big\| |\phi_t^+\rangle \big\| \big/ \big\| |\phi_t^-\rangle \big\|$ *or its reciprocal is greater than* $\mathrm{e}^{35} \simeq 1.6 \times 10^{15}$.

Using equation (25) and Girsonov's transformation (18), we can write the following SDE for Γ_t in terms of the noise W_t associated with the nonlinear equation (7):

$$\mathrm{d}\Gamma_t = \lambda X_t \big(2\langle q\rangle_t - \bar{x}_t^+ - \bar{x}_t^-\big)\,\mathrm{d}t + \sqrt{\lambda} X_t\,\mathrm{d}W_t, \tag{45}$$

with initial condition $\Gamma_0 = \ln|c_+/c_-|$. By using the expression (40) for $\langle q\rangle_t$, we can rewrite the above equation as follows:

$$\mathrm{d}\Gamma_t = \lambda X_t^2 \tanh\Gamma_t\,\mathrm{d}t + \sqrt{\lambda} X_t\,\mathrm{d}W_t. \tag{46}$$

This is the result we wanted to arrive at, and we will devote the rest of the section at analysing its physical content. To proceed further with the analysis, it is convenient to perform the following time change [14],

$$t \quad \longrightarrow \quad s_t := \lambda \int_0^t X_t^2\,\mathrm{d}t', \tag{47}$$

which allows us to describe the collapse process in terms of the dimensionless quantity s that measures its effectiveness. Using equation (43), one can solve exactly the above integral and compute s as a function of t. Such a function however cannot be inverted in order to get t from s. To this end, we use the simplified expression (44) in place of the exact formula (43) to compute the integral, an expression which, as we have seen, represents a very good approximation to the time evolution of X_t throughout the whole time during which the experiment takes place (alternatively, we may initially choose Δ_t^T in such a way that X_t evolves exactly like in (44), at least from $t = 0$ to $t = T$). Accordingly, we have

$$s \equiv s_t \simeq \frac{\lambda\hbar^2\kappa^2}{3} t^3 \simeq 2.0 \times 10^{17} (t/\mathrm{s})^3 \qquad 0 \leqslant t \leqslant T = 1\ \mathrm{s}, \tag{48}$$

$$t \equiv t_s \simeq \sqrt[3]{\frac{3}{\lambda\hbar^2\kappa^2} s} \simeq (1.7 \times 10^{-6} \sqrt[3]{s})\ \mathrm{s} \qquad 0 \leqslant s \leqslant \lambda\hbar^2\kappa^2/3 = 2.0 \times 10^{17}. \tag{49}$$

Note that, according to the above equations, the physical time t depends on s through the inverse cubic root of λ, i.e. on the inverse cubic root of the mass of the pointer; this time dependence of t on λ is important since, as we shall see, it will affect the collapse time. We do not study the functional dependence between s and t for $t \geqslant T$ since, as we shall soon see and as we expect it to be, the collapse occurs at times much smaller than T.

Written in terms of the new variable s, equation (46) reduces to

$$\mathrm{d}\Gamma_s = \tanh\Gamma_s\,\mathrm{d}s + \mathrm{d}W_s; \tag{50}$$

this equation has been already analysed in [10], using the theorems of [14]; here we report the main properties.

[11] The choice made here for the collapse parameter is different from the one made in [10]. We find this new choice, which at any rate is arbitrary, more convenient for the problem under study.

4.2.1. The collapse time. According to the definition given before, a collapse occurs when $|\Gamma_t| \geqslant 35$; it would seem then natural to define the collapse time as the time when $|\Gamma_t|$ first reaches the value 35. However, one has to face the event that $|\Gamma_t|$, after having reached such a value, immediately starts decreasing in a significant way, jeopardizing in this way the effect of the collapse. To avoid such a possibility, we proceed as follows: we will compute the time it takes for $|\Gamma_t|$ to reach a value larger than 35, let us say 50, and subsequently we will show that, after having reached such a value, the probability that it gets back to a value below 35 is negligible. In this way we can be (almost) sure[12] that, once the collapse has occurred, the superposition never reappears.

Let us then consider the time $\bar{S} = \bar{S}(\omega)$ when $|\Gamma_s|$ first reaches the value 50:

$$\bar{S} \equiv \inf\{s : |\Gamma_s| \geqslant 50\}; \tag{51}$$

of course we assume that the initial state (36) is such that $|\Gamma_0| \leqslant 35$, otherwise according to our definition (as well as for all practical purposes) it would already be a reduced state, not a physically meaningful superposition. It can be proven [10] that $\bar{S}$ is *finite* with probability 1, and that its average value and variance are given by the following expressions:

$$\mathbb{E}_{\mathbb{P}}[\bar{S}] = 50 \tanh 50 - \Gamma_0 \tanh \Gamma_0, \tag{52}$$

$$\mathbb{V}_{\mathbb{P}}[\bar{S}] = F(50) - F(\Gamma_0), \qquad F(x) = x^2 \tanh^2 x + x \tanh x - x^2. \tag{53}$$

Now, $\tanh 50 \simeq 1 - 7.4 \times 10^{-44}$ which is practically 1; let us also consider e.g. the worst case, as far as the collapse mechanism is concerned, i.e. the case in which $\Gamma_0 = 0$, which means that the micro-state is initially in a equal weighted superposition of the two eigenstates. We then have that $\mathbb{E}_{\mathbb{P}}[\bar{S}] \simeq 50$ and $\mathbb{V}_{\mathbb{P}}[\bar{S}] \simeq 50$.

$\bar{S}(\omega)$ is a random variable, so we cannot tell exactly when the collapse occurs; since however we want to be quite safe that it actually occurs, let us compute the probability that $\bar{S}$ happens to be much greater than, e.g., 10^5 times its standard deviation. By a trivial application of Čebičev's inequality we have

$$\mathbb{P}[|\bar{S} - \mathbb{E}_{\mathbb{P}}[\bar{S}]| \geqslant 10^5 \sqrt{\mathbb{V}_{\mathbb{P}}[\bar{S}]} \simeq 7.1 \times 10^5] \leqslant 10^{-10}. \tag{54}$$

We can then conclude that, at time $s = \mathbb{E}_{\mathbb{P}}[\bar{S}] + 10^5 \sqrt{\mathbb{V}_{\mathbb{P}}[\bar{S}]} \simeq 7.1 \times 10^5$, the collapse has almost certainly occurred (with probability greater than $1 - 10^{-10}$) and that it is an irreversible process (as we shall soon prove). Moving back from the effective time s to the physical times t by using equation (48), we then define the collapse time as follows:

$$T_C \simeq \sqrt[3]{\frac{3(\mathbb{E}_{\mathbb{P}}[\bar{S}] + 10^5 \sqrt{\mathbb{V}_{\mathbb{P}}[\bar{S}]})}{\lambda \hbar^2 \kappa^2}} \simeq 1.5 \times 10^{-4} \text{ s} : \tag{55}$$

the collapse occurs within a time interval smaller than the perception time of a human observer. The above formula shows also that, as expected, T_C is proportional to the inverse cubic root of the mass of the pointer (since λ is proportional to the mass): the bigger the pointer, the shorter the collapse time. With our choice for λ_0, even for a 1-g pointer the reduction occurs practically instantaneously.

It is important to note that, at time $T_C \simeq 1.5 \times 10^{-4}$ s, the distance between the two Gaussian components is approximately $X_{T_C} \simeq 1.5 \times 10^{-4}$ cm: this means that, with very high probability, the collapse occurs *before* the two components have enough time to spread out in space to form a macroscopic superposition. This means that, from the physical point of view, there is *no* collapse of the wavefunction at all, since the wavefunction always remains perfectly localized in space at any stage of the experiment. In any case, we will keep talking of collapse of the wavefunction, meaning with it simply the event $|\Gamma_t| \geqslant 35$.

[12] Here, as well as in the rest of the paper, we use 'almost sure' in the physical sense of 'with very high probability', not in the mathematical sense of 'with the possible exception of a subset of measure 0'.

4.2.2. The collapse probability. Let us call P_+ the probability that Γ_s hits the point +50 before the point −50, i.e. the probability that $|\phi_s^+\rangle$ survives during the collapse process so that the outcome of the measurement is '$+\hbar/2$'. Such a probability turns out to be equal to [10]

$$P_+ = \frac{1}{2}\frac{\tanh 50 + \tanh \Gamma_0}{\tanh 50}; \tag{56}$$

while the probability P_- that Γ_s hits the point −50 before the point +50, i.e. that the outcome of the experiment is '$-\hbar/2$', is of course:

$$P_- = \frac{1}{2}\frac{\tanh 50 - \tanh \Gamma_0}{\tanh 50}. \tag{57}$$

By taking into account that $\tanh 50 \simeq 1 - 7.4 \times 10^{-44} \simeq 1$, we can write, with extremely good approximation,

$$P_+ \simeq \frac{1}{2}[1 + \tanh \Gamma_0] = \frac{e^{\Gamma_0}}{e^{\Gamma_0} + e^{-\Gamma_0}} = \frac{e^{2\gamma_0^+}}{e^{2\gamma_0^+} + e^{2\gamma_0^-}} = |c_+|^2, \tag{58}$$

$$P_- \simeq \frac{1}{2}[1 - \tanh \Gamma_0] = \frac{e^{-\Gamma_0}}{e^{\Gamma_0} + e^{-\Gamma_0}} = \frac{e^{2\gamma_0^-}}{e^{2\gamma_0^+} + e^{2\gamma_0^-}} = |c_-|^2. \tag{59}$$

We see that the probability of getting one of the two possible outcomes is practically *equivalent to the Born probability rule!* On the one hand, this is an entirely expected results, since collapse models have been designed precisely in order to solve the measurement problem and in particular to reproduce quantum probabilities; on the other hand, it is striking that a very general equation like equation (7), which is meant to describe both quantum systems as well as macroscopic classical objects (i.e. all physical situations, at the non-relativistic level), when applied to a measurement situation, provides not only a consistent description of the measurement process, but also reproduces quantum probabilities with such a good precision.

4.2.3. Stability of the collapse process. We have already anticipated that, since Γ_s evolves randomly, there is the chance that, after having reached e.g. the value +50, i.e. after that the wavefunction collapsed to the state $|\phi^+\rangle$, it becomes smaller than 50 instead of keeping increasing, eventually getting closer and closer to 0. When such an event occurs, the superposition of the two Gaussian wavefunctions, which was previously reduced, reappears again, jeopardizing in this way the entire collapse process and localization properties of the pointer. We now give an estimate of the probability for such an event to occur.

Let us call Q_+ the probability that Γ_s, after having reached the value +50 at time $\bar{S}$, does *not* go back to a value smaller 35:

$$Q_+ := \mathbb{P}\left[\inf_{s \geqslant \bar{S}} \Gamma_s \geqslant 35\right]; \tag{60}$$

such a probability turns out to be [10]

$$Q_+ \geqslant (1 + \tanh 50)\frac{\tanh 15}{1 + \tanh 15} \simeq 1 - 9.3 \times 10^{-14}, \tag{61}$$

which is practically equal to 1: once a localization occurs, the superposition can de facto never reappear.

4.3. State vector after the collapse

At time $t \geqslant T_C$ the normalized sate vector $|\Psi_t\rangle \equiv |\Phi_t\rangle/\||\Phi_t\rangle\|$, with $|\Phi_t\rangle$ given in (38), is

$$|\Psi_t\rangle = \frac{|+\rangle \otimes |G+, t\rangle + \epsilon_t |-\rangle \otimes |G-, t\rangle}{\sqrt{1+\epsilon_t^2}}, \tag{62}$$

where $\epsilon_t \equiv \mathrm{e}^{-(\gamma_t^+ - \gamma_t^-)}$ and the normalized Gaussian states $\langle x|G\pm, t\rangle$ are defined as follows:

$$\langle x|G\pm, t\rangle = \sqrt[4]{\frac{1}{2\pi\sigma_q^2}} \exp\left[-\frac{1-i}{4\sigma_q^2}\left(x - \bar{x}_t^{\pm}\right)^2 + \mathrm{i}\bar{k}_t^{\pm} x + \mathrm{i}\theta_t^{\pm}\right]. \tag{63}$$

Let us assume that the collapse occurred in favour of the '$+\hbar/2$' eigenvalue, i.e. in such a way that $\Gamma_t \geqslant 35$ for $t \geqslant T_C$, with very high probability; it follows that

$$\epsilon_t \leqslant \mathrm{e}^{-35} \simeq 6.3 \times 10^{-16} \qquad \forall\, t \geqslant T_C, \tag{64}$$

and we can write, with excellent accuracy,

$$|\Psi_t\rangle \simeq |+\rangle \otimes |G+, t\rangle. \tag{65}$$

We recover in this way the *postulate of wave packet reduction* of standard quantum mechanics: at the end of the measurement process, the state of the micro-system is reduced to the eigenstate corresponding to the eigenvalue which has been obtained as the outcome of the measurement, the outcome being defined by the surviving Gaussian component ($|G+, t\rangle$ in this case). Note the important fact that, according to our model, the collapse acts directly only on the pointer of the measuring apparatus, not on the micro-system; however, the combined effect of the collapse plus the von Neumann type of interaction is that the microscopic superposition of the spin states of the micro-system gets rapidly reduced right after the measurement.

Note finally that, after the collapse, the states of the micro-system and of the pointer are de facto factorized: as such, after the measurement process one can, for all practical purposes, disregard the pointer and focus only on the micro-system for future experiments or interactions with other systems, as it is custom in laboratories.

4.4. The end of the experiment

In this final subsection we study how, after the collapse, the 'winning' component ($|G+, t\rangle$ or $|G-, t\rangle$) moves in space, i.e. how their centres $\bar{x}_t^+$ or $\bar{x}_t^-$ evolve in time, whether they move in such a way to display the correct outcome of the measurement. To this purpose let us define

$$\tilde{X}_t \equiv \bar{x}_t^+ + \bar{x}_t^-, \qquad \tilde{K}_t \equiv \bar{k}_t^+ + \bar{k}_t^-, \tag{66}$$

so that $\bar{x}_t^+$ and $\bar{x}_t^-$ as functions of X_t and $\tilde{X}_t$ are given by $\bar{x}_t^+ = (X_t + \tilde{X}_t)/2$ and $\bar{x}_t^- = -(X_t - \tilde{X}_t)/2$. From equations (23) and (24), taking also into account (40), one finds out that $\tilde{X}_t$ and $\tilde{K}_t$ satisfy the following SDEs:

$$\begin{aligned} \mathrm{d}\tilde{X}_t &= \frac{\hbar}{m}\tilde{K}_t\,\mathrm{d}t + \omega X_t \tanh\Gamma_t\,\mathrm{d}t + 2\sqrt{\omega}\sigma_q\,\mathrm{d}W_t, & \tilde{X}_0 &= 0\ \mathrm{m}, \\ \mathrm{d}\tilde{K}_t &= 2\lambda X_t \tanh\Gamma_t\,\mathrm{d}t + 2\sqrt{\lambda}\,\mathrm{d}W_t, & \tilde{K}_0 &= 0\ \mathrm{m}^{-1}, \end{aligned} \tag{67}$$

where X_t is given by equation (43). This is a nonlinear system, since it depends in a nonlinear way on Γ_t, which is also a stochastic process; as such (to our knowledge) the system cannot be exactly solved. To circumvent this problem, let us consider the following two auxiliary linear systems:

$$\begin{aligned} \mathrm{d}\tilde{X}_t^{\pm} &= \frac{\hbar}{m}\tilde{K}_t^{\pm}\,\mathrm{d}t \pm \omega X_t\,\mathrm{d}t + 2\sqrt{\omega}\sigma_q\,\mathrm{d}W_t, & \tilde{X}_0^{\pm} &= 0\ \mathrm{m}, \\ \mathrm{d}\tilde{K}_t^{\pm} &= \pm 2\lambda X_t\,\mathrm{d}t + 2\sqrt{\lambda}\,\mathrm{d}W_t, & \tilde{K}_0^{\pm} &= 0\ \mathrm{m}^{-1}. \end{aligned} \tag{68}$$

(with an obvious meaning of the symbols), which have been obtained in the first case (+) by replacing $\tanh\Gamma_t$ with +1, and in the second case (−) by replacing $\tanh\Gamma_t$ with −1. Clearly, we have $\tilde{X}_t^- \leqslant \tilde{X}_t \leqslant \tilde{X}_t^+$ and $\tilde{K}_t^- \leqslant \tilde{K}_t \leqslant \tilde{K}_t^+$ for any t such that $X_t \geqslant 0$, which is true for all the time during which the experiment takes place, and much longer. Such linear systems can be easily solved; concerning $\tilde{X}_t^\pm$, and after some tedious calculations one finds the following time dependence for the mean:

$$\mathbb{E}_{\mathbb{P}}\big[\tilde{X}_t^\pm\big] = \pm\begin{cases} -X_t + \hbar\kappa t & \text{for} \quad t < T, \\ -X_t + \hbar\kappa T & \text{for} \quad t \geqslant T, \end{cases} \tag{69}$$

and for the variance:

$$\mathbb{V}_{\mathbb{P}}\big[\tilde{X}_t^\pm\big] = 4\sigma_q^2\left[\omega t + \frac{(\omega t)^2}{2} + \frac{(\omega t)^3}{12}\right]. \tag{70}$$

We use the above results to approximate the time evolution of $\tilde{X}_t$ and thus of $\bar{x}_t^+$ and $\bar{x}_t^-$, which we are interested in. We consider separately the case $t \leqslant T_C$ (before the collapse) and $t \geqslant T_C$ after the collapse: in the first case, we cannot control the behaviour of Γ_t, thus the most we can say is that $|\tanh\Gamma_t| \leqslant 1$, which has already been used to bound $\tilde{X}_t$ between $\tilde{X}_t^-$ and $\tilde{X}_t^+$; in the second case, we know that with very high probability $|\tanh\Gamma_t| \geqslant \tanh 35$, which is a very strong bound.

Case 1, before the collapse: $t \leqslant T_C$. Within this time interval, the two Gaussian components $|G+, t\rangle$ and $|G-, t\rangle$ start separating, as X_t increases in time; in particular, at time $t = T_C$, when the collapse has (almost certainly) occurred, we have

$$\mathbb{E}_{\mathbb{P}}\big[\tilde{X}_{T_C}^\pm\big] \simeq \pm\tfrac{1}{2}\hbar\kappa\omega T_C^2 \simeq \pm 5.9 \times 10^{-15}\ \text{m}, \tag{71}$$

which has being obtained from equation (69) by expanding X_t, as given by equation (43), to second order in ωt; moreover, we have from equation (70):

$$\mathbb{V}_{\mathbb{P}}\big[\tilde{X}_{T_C}^\pm\big] \simeq 4\omega\sigma_q^2 T_C \simeq 6.5 \times 10^{-35}\ \text{m}^2. \tag{72}$$

This means that, on a macroscopic scale, $\tilde{X}_{T_C}^\pm \simeq \mathbb{E}_{\mathbb{P}}\big[\tilde{X}_{T_C}^\pm\big]$; since $X_{T_C} \simeq \hbar\kappa T_C \simeq 1.5 \times 10^{-6}\ \text{m} \gg \mathbb{E}_{\mathbb{P}}\big[\tilde{X}_{T_C}^\pm\big]$, and keeping in mind that $\tilde{X}_{T_C}^- \leqslant \tilde{X}_{T_C} \leqslant \tilde{X}_{T_C}^+$, we can write, with very high probability and very good approximation:

$$\bar{x}_{T_C}^+ \simeq +\tfrac{1}{2}X_{T_C} \simeq +\tfrac{1}{2}\hbar\kappa T_C \simeq +7.7 \times 10^{-7}\ \text{m}, \tag{73}$$

$$\bar{x}_{T_C}^- \simeq -\tfrac{1}{2}X_{T_C} \simeq -\tfrac{1}{2}\hbar\kappa T_C \simeq -7.7 \times 10^{-7}\ \text{m}. \tag{74}$$

Accordingly, and as expected, the two components move symmetrically in opposite directions, one towards right and the other towards left, but not fast enough for a macroscopic superposition to occur, before the collapse enters into play and suppresses one of them.

Case 2, after the collapse: $t \geqslant T_C$. Let us assume that the collapse is such that the outcome '$+\hbar/2$' is given; this means that almost certainly $\Gamma_t \geqslant 35, \forall\, t \geqslant T_C$. Given this, let us first of all show that $\tilde{X}_t$ remains very close to $\tilde{X}_t^+$, for very long times; then, by approximating $\tilde{X}_t$ with $\tilde{X}_t^+$, we will show how $\bar{x}_t^+$ and $\bar{x}_t^-$ evolve in time.

From equations (67) and (68), taking into account that $\tanh\Gamma_t \geqslant -1$, we find

$$\begin{aligned} \tilde{K}_t^+ - \tilde{K}_t \ &= 2\lambda\int_0^t X_{t'}(1 - \tanh\Gamma_{t'})\,\mathrm{d}t' \leqslant 4\lambda\int_0^t X_{t'}\,\mathrm{d}t' \\ &= \frac{8\lambda\hbar\kappa}{\omega^2}\left[1 - \mathrm{e}^{-\omega t/2}\left(\cos\frac{\omega t}{2} + \sin\frac{\omega t}{2}\right)\right] \simeq 2\lambda\hbar\kappa t^2 \end{aligned} \tag{75}$$

and

$$\tilde{X}^+_t - \tilde{X}_t = \frac{\hbar}{m}\int_0^t \left(\tilde{K}^+_{t'} - \tilde{K}_{t'}\right) \mathrm{d}t' + \omega \int_0^t X_{t'}(1 - \tanh \Gamma_{t'})\, \mathrm{d}t'$$
$$\leqslant \frac{\hbar}{m}\int_0^t \left(\tilde{K}^+_{t'} - \tilde{K}_{t'}\right) \mathrm{d}t' + 2\omega \int_0^t X_{t'}\, \mathrm{d}t' \leqslant -2(X_t - \hbar\kappa t) \simeq \hbar\kappa\omega t^2. \tag{76}$$

At time $t = T_C$, we then have $\tilde{K}^+_{T_C} - \tilde{K}_{T_C} \simeq 2\lambda\hbar\kappa T_C^2 \simeq 2.8 \times 10^{12}\ \mathrm{m}^{-1}$, and $\tilde{X}^+_{T_C} - \tilde{X}_{T_C} \simeq \hbar\kappa\omega T_C^2 \simeq 1.2 \times 10^{-14}$ m.

We use these results as initial conditions, at time T_C, to find, by integrating once more equations (67) and (68), and by using the two inequalities $\tanh \Gamma_t \geqslant \eta \equiv \tanh 35$, $\forall\, t \geqslant T_C$ and $X_t \leqslant \ell \simeq 1$ cm, the following estimates:

$$\tilde{K}^+_t - \tilde{K}_t = \tilde{K}^+_{T_C} - \tilde{K}_{T_C} + 2\lambda \int_{T_C}^t X_{t'}(1 - \tanh \Gamma_{t'})\, \mathrm{d}t' \leqslant \tilde{K}^+_{T_C} - \tilde{K}_{T_C} + 2\lambda\eta\ell(t - T_C) \tag{77}$$

and

$$\tilde{X}^+_t - \tilde{X}_t \leqslant \tilde{X}^+_{T_C} - \tilde{X}_{T_C} + \frac{\hbar}{m}\int_{T_C}^t \left(\tilde{K}^+_{t'} - \tilde{K}_{t'}\right) \mathrm{d}t' + \omega \int_{T_C}^t X_{t'}(1 - \tanh \Gamma_{t'})\, \mathrm{d}t'$$
$$\leqslant \tilde{X}^+_{T_C} - \tilde{X}_{T_C} + \frac{\hbar}{m}\left(\tilde{K}^+_{T_C} - \tilde{K}_{T_C}\right)(t - T_C) + \frac{\omega^2}{4}\eta\ell(t - T_C)^2 + \omega\eta\ell(t - T_C)$$
$$\simeq \omega\hbar\kappa T_C^2 + \frac{\omega^2}{2}\hbar\kappa T_C^2(t - T_C) + \frac{\omega^2}{4}\eta\ell(t - T_C)^2 + \omega\eta\ell(t - T_C)$$
$$\simeq (1.2 \times 10^{-14} + 2.9 \times 10^{-19} t + 5.0 \times 10^{-42} t^2)\ \mathrm{m}. \tag{78}$$

We see that for very long times, by far much longer than the time during which the experiment takes place, the distance between $\tilde{X}_{T_C}$ and $\tilde{X}^+_{T_C}$ remains small, so small that we can replace $\tilde{X}_{T_C}$ with $\tilde{X}^+_{T_C}$ for all practical purposes.

On the other hand, $\tilde{X}^+_{T_C}$ is, on a macroscopic scale, very close to its average value $\mathbb{E}_{\mathbb{P}}\left[\tilde{X}^+_t\right]$, its variance, as given by equation (70), being extremely small; accordingly we have

$$\bar{x}^+_t = \frac{X_t + \tilde{X}_t}{2} \simeq \frac{X_t + \tilde{X}^+_t}{2} \simeq \frac{X_t + \mathbb{E}_{\mathbb{P}}\left[\tilde{X}^+_t\right]}{2} = \begin{cases} +\dfrac{\hbar\kappa t}{2} & t < T, \\ +\dfrac{\hbar\kappa T}{2} & t \geqslant T, \end{cases} \tag{79}$$

which is the desired result: the pointer, represented in this case by $|G+, t\rangle$, moves at a constant speed towards the right and stops at the position $\hbar\kappa T/2$, displaying in this way the correct outcome.

To conclude the analysis, let us see what happens also to the other component, $|G-, t\rangle$, which has been suppressed by the spontaneous reduction process. Its centre $\bar{x}^-_t$ moves approximately as follows:

$$\bar{x}^-_t = -\frac{X_t - \tilde{X}_t}{2} \simeq -\frac{X_t - \tilde{X}^+_t}{2} \simeq -\frac{X_t - \mathbb{E}_{\mathbb{P}}\left[\tilde{X}^+_t\right]}{2} \simeq \begin{cases} -\dfrac{\hbar\kappa t}{2} & t < T, \\ -\dfrac{\hbar\kappa T}{2} & T \leqslant t \ll \omega^{-1}, \\ +\dfrac{\hbar\kappa T}{2} & t \gg \omega^{-1}. \end{cases} \tag{80}$$

i.e. the negligible Gaussian component moves to the left of the graduate scale, but then slowly converges towards the other wavefunction.

As a final remark, we note that, at very long times of order $\omega^{-1} \simeq 2.0 \times 10^4$ s, the statistical fluctuations become relevant also on the macroscopic scale, thus approximating any

actual value with its statistical average becomes less and less precise. However, times of order $\omega^{-1} \simeq 2.0 \times 10^4$ s are by far much longer than the time required for the experiment to end; moreover, for such long times the assumption that the global system is isolated certainly loses its validity; the measurement model should then be refined, in order to include so long time scales.

5. Conclusions

In the present work we have analysed the quantum theory of measurement within the framework of dynamical reduction models, resorting to the von Neumann-type scheme of measurement process and to the QMUPL model of spontaneous wavefunction collapse. We have proven the properties listed in the introductory section, showing in this way how the axioms 4 and 5 of standard quantum mechanics arise in quite a straightforward way from the dynamical evolution law governing models of spontaneous wavefunction collapse.

We hope that our analysis makes clearer the mechanism with which dynamical reduction models provide, at least at the non-relativistic level, such an accurate description of measurement processes, and more generally of all physical situations.

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Appendix. A mistake corrected in [10]

When $\kappa = 0$, i.e. for a free particle of mass m moving according to the SDE:

$$\mathrm{d}\psi_t(x) = \left[-\frac{\mathrm{i}}{\hbar}\frac{p^2}{2m}\,\mathrm{d}t + \sqrt{\lambda}(q - \langle q\rangle_t)\,\mathrm{d}W_t - \frac{\lambda}{2}(q - \langle q\rangle_t)^2\,\mathrm{d}t\right]\psi_t(x), \quad \text{(A.1)}$$

the two equations (25) and (26) for γ_t and θ_t, respectively, become (we neglect the $\pm$):

$$\mathrm{d}\gamma_t = \left(\lambda\bar{x}_t^2 + \frac{\hbar}{m}\alpha_t^{\mathrm{I}} + \frac{\lambda}{4\alpha_t^{\mathrm{R}}}\right)\mathrm{d}t + \sqrt{\lambda}\bar{x}_t\{\mathrm{d}\xi_t - 2\sqrt{\lambda}\bar{x}_t\,\mathrm{d}t\} \quad \text{(A.2)}$$

$$\mathrm{d}\theta_t = \left(-\frac{\hbar}{2m}\bar{k}_t^2 - \frac{\hbar}{m}\alpha_t^{\mathrm{R}} + \frac{\lambda\alpha_t^{\mathrm{I}}}{4(\alpha_t^{\mathrm{R}})^2}\right)\mathrm{d}t + \sqrt{\lambda}\frac{\alpha_t^{\mathrm{I}}}{\alpha_t^{\mathrm{R}}}\bar{x}_t\{\mathrm{d}\xi_t - 2\sqrt{\lambda}\bar{x}_t\,\mathrm{d}t\}, \quad \text{(A.3)}$$

which differ from the corresponding equations (12) and (13) of [10], in the first case for the extra factor $\lambda/4\alpha_t^{\mathrm{R}}$ and in the second case for the factor $\lambda\alpha_t^{\mathrm{I}}/4\big(\alpha_t^{\mathrm{R}}\big)^2$. We correct in this way a mistake made in [10], which however does not affect the other results contained in that paper.

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